RESEARCH STATEMENT

ARKA ADHIKARI

During the past years of my postdoctoral career, I have worked on a multitude of subjects exploring aspects of statistical physics phenomena on lattices. Beyond this, I also have experience with mean-field probability models developed in my PhD. Overall, I have developed a deep intuition in understanding multiple types of physical and probabilistic phenomena in a wide variety of models. My experience allows me to be flexible in thinking and make significant progress in various problems of interest. In what follows, I will describe some of the more important branches of my research and my future plans.

1. Lattice Gauge Theories

Some of the subjects I explored in the past few years were Yang-Mills theory and Lattice Gauge Theories. I have already written two papers in this field [2] and [4] and am currently exploring future directions. Before I describe my research in detail, I want to establish some context. Quantum Field Theory originated in physics in order to explain the origin of the fundamental forces of nature, such as electromagnetism and the strong nuclear force that holds atoms together. To describe these forces using formulas, physicists first propose some Lagrangian determining the evolution of the particles under consideration and compute an associated path integral; equivalently, one can define a Hamiltonian function and associated Hilbert Space.

There are mathematical problems with directly trying to approach Yang-Mills Theory in the path integral language of physicists. Proposed by Wilson in the 1970s [15], a mathematically formal way to approach this problem is through the language of lattice gauge theory. For a detailed list of references, please see [12] and the references therein. By performing a Wick rotation, the path integral in physics becomes a probability model; furthermore, one can regularize infinities in QFTs by computing appropriate quantities on a lattice. Given a general group G and a representation ρ of G, one can define the following lattice Yang-Mills Theories. On the lattice $\Lambda_N := [-N, N]^4$, one defines a Yang-Mills configuration to be a map from the set of edges of the lattice with orientation $E_N \to G$. A plaquette $p \in P_n$ on the lattice consists of four edges $p = (e_1, e_2, e_3, e_4)$ that wrap around as a 2-D square on the lattice.

$$H_N := \sum_{p=(e_1,e_2,e_3,e_4)\in P_N} \rho(U_{e_1}U_{e_2}U_{e_3}U_{e_4}). \tag{1.1}$$

The central objects of study in this statistical physics model are the Wilson loops W_c. Given a closed path $c = (e_1, \dots, e_n)$, $W_c := \rho(U_{e_1}U_{e_2}\dots U_{e_n})$. There are two physical properties that can be understood by Wilson Loops.

(1) Quark confinement: On the energy scale we see in the real world, quarks are not seen by themselves; instead, these quarks only exist as small parts of bigger particles such as protons or neutrons. This is called quark confinement; on a physical level,quarks are confined to be parts of bigger particles because it should take too much energy to maintain two quarks at long distances from each other. In the language of Wilson loops, if W_c is a Wilson loop surrounding a region of area A, then one should have $\mathbb{E}[W_c] \approx \exp[-CA]$ for some constant C. When quark confinement does not hold, W_c should satisfy the perimeter law $W + c \approx \exp[-CP]$, where P is the perimeter of loop c.

(2) Mass Gap: Gluons, the particles that facilitate the strong interaction, should conglomerate into larger particles called glueballs. These particles should have a spontaneously generated mass. In the language of Wilson loops, this translates into correlation decay. Namely, one wants to show that $-\frac{1}{d}\log(\mathbb{E}[W_{c_1}W_{c_2}] - \mathbb{E}[W_{c_1}]\mathbb{E}[W_{c_2}]) > 0$, where c_1 and c_2 are two Wilson loops separated by a distance d.

My recent papers [2] and [4] address both of these questions in the context of finite non-Abelian groups G and Yang-Mills models coupled to a Higgs boson; the Higgs boson is responsible for the spontaneous generation of mass in many physical models. In lattice gauge theories, it can be expressed as a map from the vertices of the lattice to some group of interest.

The following theorems were the main result of my paper [2].

Theorem 1.1 (Informal Version of Theorems 2.1 and 2.3)**.** *For sufficiently large β and κ, the main order contributions to the Wilson loop observables are by the number of minimal vortices (roughly, those plaquette excitations, or nontrivial gauge configurations, centered around a single edge with $\sigma_e \neq 1$) that are centered around the edges of γ. Furthermore, the number of minimal vortex excitations along the edges of γ can be treated as roughly a Poisson random variable with parameter $O(|\gamma|)$.*

Define the quantity

$$A_{\beta,\kappa} := \frac{\sum_{g\neq 1} \rho(g) \exp[12\beta Re[(\rho(g) - \rho(1))]] \exp[2\kappa Re[\rho(g) - \rho(1)]]}{\sum_{g\neq 1} \exp[12\beta Re[(\rho(g) - \rho(1))]] \exp[2\kappa Re[\rho(g) - \rho(1)]]}$$

$$\mathbb{E}[W_\gamma] \approx \mathbb{E}[A^X_{\beta,\kappa}], \tag{1.2}$$

where X is a Poisson random variable with expectation $|\gamma| \sum_{g\neq 1} \exp[12\beta Re[\rho(g)-\rho(1)] \exp[2\kappa Re[\rho(g)-\rho(1)]]$.

Theorem 1.2 (Based on Theorems 4.1, 4.2 and 5.4)**.** *For β sufficiently large and κ sufficiently small, the main order contribution to Wilson loop expectations comes from minimal vortices centered around the edges of γ .*

One can compute the Wilson loop expectation as,

$$\mathbb{E}[W_\gamma] \approx \mathbb{E}[Tr[\hat{A}^X_{\beta,\kappa}]], \tag{1.3}$$

where $\hat{A}_{\beta,\kappa}$ is a matrix defined in Theorem 5.2 and X is a Poisson random variable whose expectation is $O(|\gamma|)$.

In layperson's terms, the above theorem shows that a discrete non-Abelian lattice gauge theory with a Higgs boson satisfies a perimeter law decay at large values of β; it is believed by physicists that large β is the most physically relevant regime for Yang-Mills Theory. Furthermore, considering non-Abelian gauge groups adds a significant difficulty to the analysis. In Abelian groups, it is relatively straightforward to decompose configurations U into more simple configurations $U = U_1U_2$ when appropriate. For example, if a configuration U is supported (or behaves non-trivially) on two distinct regions S_1 and S_2, then one can let $U_1 = U|_{S_1}$, the restriction of U to S_1, and $U_2 = U|_{S_2}$. Under this choice, $U = U_1U_2$ and this decomposition can allow one to deduce many properties, such as correlation decay.

When one treats non-Abelian models, such a natural decomposition no longer exists; namely, it is no longer clear that if a configuration U is supported on two regions S_1 and S_2 that are disjoint, then there exists a natural splitting $U = U_1U_2$. In fact, there are topological obstructions to such a splitting: if S_1 and S_2 are two loops that are merely interlinked with each other but not touching, then there is no way to split the configuration U into a configuration supported on U_1 and one supported on U_2. This suggests that configurations could be correlated with each other via long distances through this phenomenon of interlinking. Furthermore, such long-range interactions could lead to complicated effects on the expectations of Wilson loop variables. The impact of these interactions is made even further unclear with the presence of Higgs bosons.

My main theorem in [2] was able to accurately control the effect of these long-range interactions and derive a perimeter law decay for Wilson loop expectations at large β. In the work [4] with S. Cao, we increase our understanding of the interactions and derive the following result.

Theorem 1.3. *Let $\beta \geq \frac{1}{\Delta_G}(114 + 4\log|G|)$. Let $L \geq 0$. Let $B_1, B_2 \subset \Lambda$ be rectangles that are at a ℓ^∞ distance at least L from each other (i.e., the ℓ^∞ distance between any vertex x of B_1 and any vertex y of B_2 is at least L). Let $k_1, k_2 \geq 1$, and let $f_1 : G^{k_1} \to \mathbb{C}, f_2 : G^{k_2} \to \mathbb{C}$ be conjugacy invariant functions. For $i = 1, 2$, let $\gamma^{(i)}_1, \ldots, \gamma^{(i)}_{k_i}$ be closed loops contained in B_i. Let $\Sigma \sim \mu_{\Lambda,\beta}$. Then*

$$|\mathrm{Cov}(f_1(\Sigma_{\gamma^{(1)}_1}, \ldots, \Sigma_{\gamma^{(1)}_{k_1}}), f_2(\Sigma_{\gamma^{(2)}_1}, \ldots, \Sigma_{\gamma^{(2)}_{k_2}}))| \leq 4(4 \cdot 10^{24}|G|^2)^{|B_1|+|B_2|}\|f_1\|_\infty\|f_2\|_\infty e^{-(\beta/2)\Delta_G(L-1)}.$$

This establishes decay of correlations for discrete non-Abelian lattice gauge theories with Higgs boson; namely, it proves the existence of a mass gap on this model of a Yang-Mills theory. In the future, I plan to extend the analysis of these results to the study of continuous Lie groups.

2. KPZ UNIVERALITY

Another subject I am interested in researching is the KPZ equation and universality. The KPZ equation given by,

$$\partial_t f(t,x) = \partial_x^2 f(t,x) + \beta(\partial_x f(t,x))^2 + \xi(t,x)$$

was initially created by physicists in order to describe the growth of surfaces in random media. Here, $\xi(t,x)$ formally represents a Gaussian white noise at the point (t,x). For example, physicists believe that $f(t,x)$ should describe the height of ballistic deposition; in this process, particles randomly fall onto the points of a lattice. If a particle is falling at position x, it either sticks to the top of one of the columns next to it at positions $x-1$ or $x+1$, whichever is larger, or on top of the column at x. After an appropriate scaling in space and time, the height of this procedure should satisfy the KPZ equation.

The most surprising part of the KPZ equation is its ubiquity; various models from disparate subjects, such as integrable probability and random matrix theory, are phenomena that are governed by the KPZ equation or have distributions that appear in the KPZ equation. For example, the fluctuation of the largest eigenvalue of a random Wigner matrix satisfies the Tracy-Widom law; this Tracy-Widom law also describes the pointwise fluctuation of the KPZ equation. It is of primary interest to understand this universality phenomenon of the KPZ equation. However, difficulties with dealing with the nonlinearity appearing in the KPZ equation have limited many approaches to understanding the KPZ equation. The most common approaches to understanding the KPZ equation involve using integrable methods [9] to generate exact formulae; rough path theory, as detailed in the works by Hairer [14], allows one to rigorously construct the solution of the KPZ equation, but it requires heavy machinery.

The most well-studied models on the lattice are the polymers, whose partition functions are written as follows

$$Z = \frac{1}{2^n} \sum_{S \in W_n} \exp[\beta \sum_{(t,x)\in S} \omega_{(t,x)}]$$

and with free energy given by $f := \log Z$. This model corresponds to the following driven process: consider the function

$$\phi^{\text{poly}}(v_1,v_2) := \frac{1}{\beta} \log \left[\frac{\exp[\beta v_1] + \exp[\beta v_2]}{2} \right].$$

Then, $f(x,t) = \phi^{\text{poly}}(f(x-1,t-1), f(x+1,t-1)) + y(x,t)$, where $y(x,t)$ is some independent noise at each lattice point. The key reason that the polymer models are tractable is that they have explicit formulas that can be analyzed. In the paper [9], which showed the convergence of the free energy of the rescaled polymer to the KPZ equation, the authors used a chaos expansion of the polymer as a function of the noise. With these formulas in hand, one can derive explicit weak convergence of the polymer to the solution of the KPZ equation.

However, such an analysis will completely break down with even minor modifications of the driving function, even though mathematicians would conjecture that the KPZ equation would be the unique scaling limit even with significantly different driving functions, such as a max function used in ballistic deposition. It is of crucial interest to be able to understand the universality phenomenon of the KPZ equation in a tractable manner without appeal to integrability. In the paper [5] with S. Chatterjee, we have obtained a method to establish the universality of the KPZ equation in a large set of models. The following is our main Theorem,

Theorem 2.1 (Informal Version of Theorem 1.1)**.** *Define f_N as $f_N(x,t) = \phi(\tilde{f}_N(x-1,t-1), \tilde{f}_N(x+1,t-1)) + y(x,t)$ where ϕ is symmetric, equivariant under translation, and has sufficiently many derivatives. Furthermore, suppose that $\partial_1^2 \phi := \beta \neq 0$. Define $\tilde{f}_N$ to be the renormalized version,*

$$\tilde{f}_N(x,t) := f_N(\sqrt{N}x, Nt) - \left(V + \frac{1}{2}\beta N^{1/2}\mu_2 + \frac{1}{6}\beta^2 N^{1/4}\mu_3 + \frac{1}{24}\beta^3(\mu_4 - 3\mu_2^2) + N\psi(0,0) \right) t, \tag{2.1}$$

where the μ_i are the moments of the random variable $y(x,t)$.

Then the $C(\mathbb{R}\times\mathbb{R}_+)$-valued random function $\exp(\beta\tilde{f}_N)$ converges in law as $N\to\infty$ to a solution $\mathcal{Z}$ of the stochastic heat equation with multiplicative noise

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_x^2 \mathcal{Z} + \sqrt{2\mu_2}\beta\mathcal{Z}\xi, \quad \mathcal{Z}(0,\cdot)\equiv 1, \tag{2.2}$$

where ξ is standard space-time white noise.

By deriving an error propagation equation along with defining appropriate renormalization terms to characterize macroscopic differences, the paper [5] was able to establish the universality of the KPZ equation under a broad family of driving functions. This, too, was done without appealing to heavy tools like rough path theory. My future work in this direction involves understanding detailed properties of the KPZ fixed point using a similar technique.

3. Random Walks

The study of random walks is a classical subject in probability, and many of its fundamental results have far-reaching applications to multitudes of other topics. For an example from my own research, the Hamiltonian of an $SU(n)$ lattice gauge theory can be locally approximated by a Gaussian free field. The correlation structure of this model is given by the Green's function of a simple random walk. A deeper understanding of random walks can lead to progress in a variety of other fields.

In works with I. Okada, [7, 8], we explored the large deviation behavior of the capacity of the range of a random walk. Given a set A the capacity of A is defined as,

$$\text{Cap}(A) = \sum_{a\in A}\mathbb{P}(R'_a \cap A = \emptyset)$$

Here, R'_a is a random walk that starts at the point a (and the intersection is understood as being after time 0). The capacity can be understood as an appropriate notion for the size of a set in higher dimensions, and , in the context of random walks, it provides information on detailed properties of the intersection of random walks.

There are works that establish laws of large numbers and central limit theorems for the capacity of the range of a random walk, [11, 10], but the precise large deviation behavior is far more difficult to understand. Rough bounds can derive some aspects of the large deviation behavior [1], but if one is interested in optimal large deviation constants, one needs to get exact asymptotics for the moments. In many instances, one can obtain large deviation constants by estimating the higher moments. However, there are multiple obstructions to obtaining tractable expressions for these higher moments in

The capacity of the range of the random walk has a decomposition of the form,

$$\text{Cap}(R[0,2n]) = \text{Cap}(R[0,n]) + \text{Cap}(R[n+1,2n]) - \chi(R[0,n],R[n+1,2n]) + \epsilon_n. \tag{3.1}$$

Namely, one can determine the capacity of a random walk by looking at the capacity of the first and second halves, and subtracting a term that represents the 'joint capacity,' which is given by χ, along with a negligible error term ϵ_n. The lower tail of the large deviation is primarily determined by the behavior of the interaction term χ, which can be expressed for general sets A and B as,

$$\chi(A,B) = \sum_{a\in A}\sum_{b\in B}\mathbb{P}(R'_a\cap A=\emptyset)G(a-b)\mathbb{P}((R')_b\cap(A\cup B)=\emptyset) \tag{3.2}$$

When trying to compute very high moments, one runs into the following difficulties. First of all, if we let A and B be the two halves of a random walk, the term $G(a-b)$ suggests that, instead of being allowed to treat the two halves of the random walk as if they were independent when taking expectations, one would have to deal with intricacies in the correlation between the two random walks. This difficulty is only compounded by the fact that the expression is not symmetric in the sets A and B; thus, tools like the Feynman-Kac formula, which would require symmetry between the two halves of the random walk, are no longer available. Finally, a precise understanding of the probability term $\mathbb{P}(R'_a\cap A=\emptyset)$ suggests that there could be effects due to long-range correlations inside the random walk.

The main result of [8] is the following,

Theorem 3.1. *Assume $b_n \to \infty$ and $b_n = O((\log\log n))$. For $d = 4$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log P(Cap(R[0,n]) - \mathbb{E}[Cap(R[0,n])] \le -\frac{\lambda n}{(\log n)^2} b_n) = -I_4(\lambda),$$

where

$$I_4(\lambda) = 4\frac{8^2}{\pi^4}\tilde{K}^{-4}(4,2)\lambda.$$

and $\tilde{K}(4,2)$ is the optimal constant in the modified Gagliardo-Nirenberg inequality, given by

$$\left[\int_{\mathbb{R}^4} g^2(x)G(x-y)g^2(y)\right]^{1/4} \le \tilde{K}(4,2)||g||_{L^2}^{1/2}||\nabla g||_{L^2}^{1/2}$$

and G here represents the Green's function of a Brownian motion.

Proving this result involved a careful decomposition of χ into its main order contribution, which we ensured was symmetric and could be written in a way to separate the two halves of the random walks in expressions of the higher moments and various lower-order terms. By ensuring that our expression for the main order term was symmetric and separated the random walks, we gained access to various tools that allowed us to get exact estimates of the higher moments. Furthermore, a monotonicity property of the probability term $\mathbb{P}(R'_a \cap A = \emptyset)$ allowed us to limit the extent to which it leads to long-range correlations in the expressions for higher moments. Beyond just being a significant technical achievement, our result deepens the conjectured link between the behavior of the capacity in d dimensions and the range of a random walk in $d-2$ dimensions. In the future, I plan to use the techniques developed in this paper to further explore detailed aspects of the behavior of random walks and apply them to answer questions in other fields of interest, like polymers in higher dimensions.

4. Mean Field Random Systems: Random Matrices and Spin Glasses

In my postdoctoral career, I focused mainly on exploring phenomena that can be found in discrete systems; however, in my PhD research, I studied mean-field systems derived from random matrices and spin glasses and have made multiple contributions to these fields. In what follows, I will describe some of my recent results in these subjects. With C. Brennecke, C. Xu, and H.T. Yau, we have established a result on the spectral gap for p-spin glass models. The main result of [3] is the following,

Theorem 4.1 (Informal Version)**.** *Define the Hamiltonian H_N as*

$$H_N = \sum_{p=2}^{\infty} \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} g_{i_1,\dots,i_p}\sigma_{i_1}\dots\sigma_{i_p},$$

where the $g_{i_1,\dots,i_p}$ are independent standard normal random variables. We let $\langle\cdot\rangle$ denote Gibbs' expectation with respect to this Hamiltonian. With the Dirichlet form $D(f) = \sum_{i=1}^N \langle(\partial_i f)^2\rangle$, where ∂_i denotes the discrete derivative with respect to the ith coordinate, then the spectral gap constant is defined as,

$$a_{H_N} := \sup_{\substack{f:\Sigma_N\to\mathbb{R},\\ f\neq const.}} \frac{\langle f; f\rangle}{D(f)}.$$

Now, for every $\epsilon > 0$, if $\sum_{p\ge 2}\sqrt{p^3\log p}\,\beta_p$ is sufficiently small (depending on ϵ), then there exist constants $c, C > 0$, independent of $N \in \mathbb{N}$, s.t.

$$\mathbb{P}\big(\{a_{H_N} > 1+\epsilon\}\big) \le C\exp(-c\,N).$$

,

The size of the spectral gap determines the rate of convergence of an appropriate Markov chain (here, the Glauber Dynamics) to equilibrium. Approaches to proving a spectral gap inequality were previously only restricted to the SK model [13]; these analyses specifically required that one was only dealing with two-spin interactions. Deriving a spectral gap inequality for p-spin models required a completely different approach from these previous results.

Concerning random matrices, one of my recent results involved investigating local laws for dynamically defined unitary matrices. The natural analog of the Dyson-Brownian motion on Unitary matrices is given by,

$$\mathrm{d}U_t = \mathrm{i}U_t \mathrm{d}W_t - \frac{1}{2}U_t \mathrm{d}t. \tag{4.1}$$

The eigenvalues of U_t are restricted to lie on the circle and can be represented as $\exp[\mathrm{i}\theta_1], \dots, \exp[\mathrm{i}\theta_N]$. For times $t < 4$, the eigenvalues will not wind around the circle, and they can be well-ordered. However, at time $t = 4$, the eigenvalues will begin to approach each other from both sides of the circle; if one tries to compute the eigenvalue spacing statistics at this time, one will find that they follow a new type of statistics called cusp statistics. This is in contrast to the edge and bulk statistics that more commonly appear in random matrices.

With B. Landon, [6] we derived the following result on the rigidity of the eigenvalue statistics for this dynamic on unitary matrices. We present the result for the cusp that forms at time $t = 4$; there are similar results for the edge and for the bulk at earlier times, but they are technically easier to prove.

Theorem 4.2. *Let $\delta > 0$ and $\varepsilon > 0$. For $N^{-1/2+\delta} \leq 4 - t \leq 10^{-1}$ we have that the following estimates hold with overwhelming probability. Uniformly for all i satisfying $1 \leq i \leq N(4-t)^2$ we have,*

$$|\theta_i(t) - \gamma_i(t)| \leq N^{\varepsilon}(4-t)^{1/6}\frac{1}{N^{2/3}i^{1/3}} \tag{4.2}$$

and for $N(4-t)^2 \leq i \leq N/2$ we have,

$$|\theta_i(t) - \gamma_i(t)| \leq N^{\varepsilon}\frac{1}{N^{3/4}i^{1/4}}. \tag{4.3}$$

Analogous estimates hold for indices i near N.

5. Future Goals

During my graduate school career, I worked on multiple problems derived from studying mean-field phenomena, such as problems arising from random matrices and spin glasses. During my postdoctoral career, I further expanded my skill set to solve problems arising from studying probabilistic and physical phenomena on discrete lattices, such as Lattice Gauge Theories, KPZ universality, and random walks. By working on these problems, I have developed a robust skill set and built deep intuition that would allow me to make significant progress in a wide variety of fields. I plan to continue exploring these fields, and I believe that the ability that I have gained from working on these problems will allow me to make significant progress in the future.

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Department of Mathematics, Stanford University
Email address: arkaa@stanford.edu